Preliminaries

We begin by recalling some basic notions of functional analysis. A measurable function f belongs to the Lebesgue space $L_p\left(\mathbb{R}\right), 1 \leq p < \infty$ if

Equation:

$$||f||_p = (\int_{-\infty}^{+\infty} |f\left(x\right)|^p dx)^{1/p} < \infty.$$

A Hilbert space is a space where an inner product is defined. In particular, the space $L_2\left(\mathbb{R}\right)$ is a Hilbert space, where the inner product of 2 functions f and g is defined as:

Equation:

$$< f, g >= \int_{-\infty}^{+\infty} f\left(x\right) g(x) dx.$$

In this presentation we work with functions defined on $\mathbb{R}$, but that take values in $\mathbb{C}[0.1ex]0.05em1.25ex$. Hence $g(x)$ denotes the complex conjugate of $g(x)$. We say that 2 functions are orthogonal if their inner product is zero. A function is Hölder continuous of order α, $(0 < \alpha \leq 1)$ at point x if :

Equation:

$$|f\left(x\right) - f\left(x + h\right)| = O\left(h^\alpha\right).$$

Multiresolution analysis

The scaling function and the subspaces

There are two ways to introduce wavelets: one is through the continuous wavelet transform, and the other is through multiresolution analysis (MRA), which is the presentation adopted here. Here we start by defining multiresolution analysis and thereafter we give one example of such MRA.

Definition

Note: (Multiresolution analysis) A multiresolution analysis of $L_2(\mathbb{R})$ is defined as a sequence of closed subspaces $V_j \subset L_2(\mathbb{R}), j \in \mathbb{Z}$ with the following properties:

1. **Equation:**

$$... \subset V_{-1} \subset V_0 \subset V_1 ...$$

2. The spaces V_j satisfy
   Equation:

$$\bigcup_{j \in \mathbb{Z}} V_j \text{ is dense in } L_2(\mathbb{R}) \text{ and } \bigcap_{j \in \mathbb{Z}} V_j = \{0\}$$

3. If $f(x) \in V_0, f\left(2^j x\right) \in V_j$. This property means that all the spaces V_j are scaled versions of the central space V_0.
4. If $f \in V_0, f(. - k) \in V_0, k \in \mathbb{Z}$. That is, V_0(and hence all the V_j) is invariant under translation.
5. There exists $\phi \in V_0$ such that $\{\phi_{0,n}; n \in \mathbb{Z}\}$ is an orthonormal basis in V_0.

Condition 5 in [link] seems to be quite contrived, but it can be relaxed (i.e.,instead of taking orthonormal basis, we can take Riesz basis). We will use the following terminology: a **level** of a multiresolution analysis is one of the V_j subspaces and one level is **coarser** (respectively **finer**) with respect to another whenever the index of the corresponding subspace is smaller (respectively bigger).

Consequence of the definition

Let us make a couple of simple observations concerning this definition. Combining the facts that

1. **Equation:**

$$\phi\left(x\right)\in V_0$$

2. $\{\phi(.-k),k\in \mathbb{Z}\}$ is an orthonormal basis for V_0
3. $\phi\left(2^jx\right)\in V_j$,

we obtain that, for fixed j, $\left\{\phi_{j,k}\left(x\right)=2^{j/2}\phi\left(2^jx-k\right),k\in \mathbb{Z}\right\}$ is an orthonormal basis for V_j.

Since $\phi\in V_0\subset V_1$, we can express ϕ as a linear combination of $\{\phi_{1,k}\}$:
Equation:

$$\begin{aligned}\phi(x) &= \sum_k h_k\phi_{1,k}\left(x\right)\\ &= \sqrt{2}\sum_k h_k\phi\left(2x-k\right).\end{aligned}$$

[link] is called the **refinement equation**, or the **two scales difference** equation. The function $\phi(x)$ is called the **scaling function**. Under very general condition, ϕ is uniquely defined by its refinement equation and the normalisation
Equation:

$$\int_{-\infty}^{+\infty} \phi(x)dx = 1.$$

The spaces V_j will be used to approximate general functions (see an example below). This will be done by defining appropriate projections onto these spaces. Since the union of all the V_j is dense in $L_2(\mathbb{R})$, we are guaranteed that any given function of L_2 can be approximated arbitrarily close by such projections, i.e.:
Equation:

$$\lim_{j \to \infty} \mathscr{P}_j f = f,$$

for all f in L_2. Note that the **orthogonal** projection of f onto V_j can be written as:
Equation:

$$\mathscr{P}_j f = \sum_{k \in \mathbb{Z}} \alpha_k \phi_{jk}.$$

where $\alpha_k =< f, \phi_{j,k} >$.

Example

The simplest example of a scaling function is given by the Haar function:
Equation:

$$\phi(x) = 1\!\!1_{[0,1]} = \begin{cases} 1 & \text{if } 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Hence we have that
Equation:

$$\phi(2x) = \begin{cases} 1 & \text{if } 0 \leq x \leq 1/2 \\ 0 & \text{otherwise} \end{cases}$$

and
Equation:

$$\phi(2x-1) = \begin{cases} 1 & \text{if } 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

The function ϕ generates, by translation and scaling, a multiresolution analysis for the spaces V_j defined by:
Equation:

$$V_j = \{f \in L_2(\mathbb{R}); \forall k \in \mathbb{Z}, f|_{[2^j k, 2^j(k+1)[} = \text{constant}\}$$

The wavelet function and the detail spaces Wj

The detail space Wj

Rather than considering all our nested spaces V_j, we would like to code only the information needed to go from V_j to V_{j+1}. Hence we define by W_j the space complementing V_j in V_{j+1} :
Equation:

$$V_{j+1} = V_j \oplus W_j$$

This space W_j answers our question: it contains the “detail” information needed to go from an approximation at resolution j to an approximation at resolution $j+1$. Consequently, by using recursively the [link], we have:
Equation:

$$\oplus_{j \in \mathbb{Z}} W_j = L_2(\mathbb{R}).$$

The main interest of MRA lies in the fact that, whenever a collection of closed subspaces satisfies the conditions in [link], there exists an orthonormal wavelet basis, noted $\{\psi_{j,k}, k \in \mathbb{Z}\}$ of W_j. Hence, we can say in very general terms that the projection of a function f onto V_{j+1} can be decomposed as

Equation:

$$\mathscr{P}_{j+1} f = \mathscr{P}_j f + \mathscr{Q}_j f,$$

where $\mathscr{P}_j$ is the projection operator onto V_j, and $\mathscr{Q}_j$ is the projection operator onto W_j. Before proceeding further, let us state clearly what the term “orthogonal wavelet ” involves.

Recapitulation: orthogonal wavelet

We introduce above the class of orthogonal wavelets. Let us define this precisely.

An orthogonal wavelet basis is associated with an orthogonal multiresolution analysis that can be defined as follows. We talk about **orthogonal** MRA when the wavelet spaces W_j are defined as the orthogonal complement of V_j in V_{j+1}. Consequently, the spaces W_j, with $j \in \mathbb{Z}$ are all mutually orthogonal, the projections $\mathscr{P}_j$ and $\mathscr{Q}_j$ are orthogonal, so that the expansion

Equation:

$$f\left(x\right) = \sum_j \mathscr{Q}_j f\left(x\right)$$

is orthogonal. Moreover, as $\mathscr{Q}_j$ is orthogonal, the projection onto W_j can explicitely be written as:

Equation:

$$\mathcal{Q}_j f(x) = \sum_k \beta_{jk} \ \psi_{jk}(x),$$

with
Equation:

$$\beta_{jk} = < f, \psi_{jk} >.$$

A sufficient condition for a MRA to be orthogonal is:
Equation:

$$W_0 \perp V_0,$$

or $< \psi, \phi(. - l) >= 0$ for $l \in \mathbb{Z}$, since the other conditions simply follow from scaling.

An orthogonal wavelet is a function ψ such that the collection of functions $\{\psi(x - l) | l \in \mathbb{Z}\}$ is an orthonormal basis of W_0. This is the case if $< \psi, \psi(. - l) >= \delta_{l,0}$.

In the following, we consider only orthogonal wavelets. We now outline how to construct a wavelet function $\psi(x)$ starting from $\phi(x)$, and thereafter we show what this construction gives with the Haar function.

How to construct ψ(χ) starting from ρ(χ)?

Suppose we have an orthonormal basis (ONB) $\{\phi_{j,k}, k \in \mathbb{Z}\}$ for V_j and we want to construct ψ_{jk} such that

- $\psi_{jk}, k \in \mathbb{Z}$ form an ONB for W_j
- **Equation:**

$$V_j \perp W_j, \text{i.e.} < \phi_{jk}, \psi_{jk'} >= 0 \ \forall k, k'$$

- **Equation:**

$$W_j \perp W_{j'} \text{ for } j \neq j'.$$

It is natural to use conditions given by the MRA aspect to obtain this. More specifically, the following relationships are used to characterize ψ :

1. Since $\phi \in V_0 \subset V_1$, and the $\phi_{1,k}$ are an ONB in V_1, we have:
   Equation:

   $$\phi\left(x\right) = \sum_k h_k \phi_{1,k}, h_k = < \phi, \phi_{1,k} >, \sum_{k \in \mathbb{Z}} \left|h_k\right|^2 = 1.$$

   (refinement equation)
2. **Equation:**

   $$\delta_{k,0} = \int \phi\left(x\right)\phi(x-k)dx$$

   (orthonormality of $\phi(.-k)$)
3. Let us now characterize $W_0 : f \in W_0$ is equivalent to $f \in V_1$ and $f \perp V_0$. Since $f \in V_1$, we have:
   Equation:

   $$f = \sum_n f_n \phi_{1,n}, \text{with} f_n = < f, \phi_{1,n} >.$$

   The constraint $f \perp V_0$ is implied by $f \perp \phi_{0,k}$ for all k.
4. Taking the general form of $f \in W_0$, we can deduce a candidate for our wavelet. We then need to verify that the $\psi_{0,k}$ are indeed an ONB of W_0.

In fact, in our setting, the conditions given above can be re-written in the Fourier domain, where the manipulations become easier (for details, see [link], chapter 5). Let us now state the result of these manipulations.

Note:(Daubechies, chap 5) If a ladder of closed subspaces $(V_j)_{j\in\mathbb{Z}}$ in $L_2(\mathbb{R})$ satisfies the conditions of the [link], then there exists an associated orthonormal wavelet basis $\{\psi_{j,k}; j,k\in\mathbb{Z}\}$ for $L_2(\mathbb{R})$ such that:

Equation:

$$\mathscr{P}_{j+1} = \mathscr{P}_j + \sum_k <.,\psi_{j,k}> \psi_{j,k}.$$

One possibility for the construction of the wavelet ψ is to take

Equation:

$$\psi(x) = \sqrt(2)\sum_n (-1)^n h_{1-n}\phi(2x-n)$$

(with convergence of this serie is L_2 sense).

Example(continued)

Let us see what the recipe of [link] gives for the Haar multiresolution analysis. In that case, $\phi(x)=1$ for $0\le x\le 1, 0$ otherwise. Hence:

Equation:

$$h_n = \sqrt{2}\int dx\phi(x)\phi(2x-n) = \begin{cases} 1/\sqrt{2} & \text{if } n=0,1 \\ 0 & \text{otherwise} \end{cases}$$

Consequently,

Equation:

$$\begin{array}{rcl} \psi(x) & = & \sqrt{2}h_1\phi\left(2x\right) - \sqrt{2}h_0\phi\left(2x-1\right) \\ & = & \phi(2x) - \phi(2x-1) \\ & = & \begin{array}{ll} 1 & \text{if } 0 \leq x < 1/2 \\ -1 & \text{if } 1/2 \leq x \leq 1 \\ 0 & \text{otherwise} \end{array} \end{array}$$

Homogeneous and inhomogeneous representation of

Inhomogeneous representation

If we consider a first (coarse) approximation of $f \in V_0$, and then "refine" this approximation with detail spaces W_j, the decomposition of f can be written as:
Equation:

$$\begin{array}{rcl} f & = & \mathscr{P}_0 f + \sum_{j=0}^{\infty} \mathscr{Q}_j f \\ & = & \sum_k \alpha_k \phi_{0,k}\left(x\right) + \sum_{j=0}^{\infty}\sum_k \beta_{j,k}\psi_{j,k}\left(x\right), \end{array}$$

where $\alpha_k =< f, \phi_{0,k} >$ and $\beta_{j,k} =< f, \psi_{j,k} >$. In this case, we talk about **inhomogeneous** representation of f.

Homogeneous representation

If we use the fact that
Equation:

$$\oplus_{j\in\mathbb{Z}} W_j \text{ is dense in } L_2\left(\mathbb{R}\right),$$

we can decompose f as a linear combination of functions $\psi_{j,k}$ only:
Equation:

$$f(x) = \sum_{j=-\infty}^{+\infty} \sum_{k} \beta_{j,k} \psi_{j,k}(x).$$

We then talk about **homogeneous** representation of f.

Properties of the homogeneous representation

- Each coefficient $\beta_{j,k}$ in [link] depends only locally on f because
  Equation:

  $$\beta_{j,k} = \int f(x) \psi_{j,k}(x) dx,$$

  and the wavelet $\psi_{j,k}(x)$ has (essentially) bounded support.
- $\beta_{j,k}$ gives information on scale 2^{-j}, near position $2^{-j}k$
- a discontinuity in f only affects a small proportion of coefficients– a fixed number at each frequency level.

The filtered transform

Introduction

We saw in a previous section an example of a scaling function ϕ, and we outlined how to construct ψ(also called the mother wavelet) starting from ϕ (the father wavelet). Suppose now we have at our disposal $\{\phi_{j,k}\}$ and $\{\psi_{j,k}\}$. In fact, it is sufficient for our purpose to know the value of these functions at dyadic points $2^{-j}k, j\in\mathbb{Z}, k\in\mathbb{Z}$. We would like to compute in an efficient way the wavelet representation described in Homogeneous and inhomogeneous representation of f, that is, we would like to have a fast algorithm to compute the wavelet coefficients.

Filter algorithm-Fast wavelet transform

We will still use the relationship between the functions spaces V_j and W_j to find a fast wavelet transform (FWT). We start by recalling that, since both the scaling function $\phi\in V_0$ and the wavelet $\psi\in W_0$ are in V_1, and since V_1 is generated by $\phi_{1,k}=\sqrt{2}\phi\left(2x-k\right), k\in\mathbb{Z}$, there exist two sequences $\{h_k\}$ and $\{g_k\}\in l^2$ such that

Equation:

$$\phi\left(x\right)=\sqrt{2}\sum_k h_k\phi\left(2x-k\right)$$

Equation:

$$\psi\left(x\right)=\sqrt{2}\sum\nolimits_k g_k\phi\left(2x-k\right)$$

for all $x\in\mathbb{R}$. On the other hand, we know that $V_1=V_0\oplus W_0$ and as we consider the orthogonal case, it follows immediately that :

Equation:

$$\sqrt{2}\phi\left(2x\right)=\sum_k\left[h_{-2k}\phi\left(x-k\right)+g_{-2k}\psi\left(x-k\right)\right]$$

Equation:

$$\sqrt{2}\phi\left(2x-1\right)=\sum_k \left[h_{1-2k}\phi\left(x-k\right)+g_{1-2k}\psi\left(x-k\right)\right].$$

These two equations [link] and [link] can be combined into a single formula:

Equation:

$$\sqrt{2}\phi\left(2x-l\right)=\sum_k \left[h_{l-2k}\phi\left(x-k\right)+g_{l-2k}\psi\left(x-k\right)\right], l\in \mathbb{Z},$$

which is called the “decomposition relation” of ϕ and ψ.

Note that, in the bi-orthogonal case there are four sequences in l^2 instead of two (denoted here by $\{h_k\}$ and $\{g_k\}$): we have two sequences for the 2-scales relations [link], [link], and two others for the decomposition relations [link], [link]. In the following algorithm, we drop the normalisation constant. Suppose that we want to decompose f as a sum of wavelets and that we have computed, or are given, the inner products of f with $\phi_{J,k}$, where J is the finest scale we can work on. We denote these inner products by c^J. Now, our task is to compute c_k^j and d_k^j, $j<J$, where

Equation:

$$\mathcal{P}_j f=\sum_k c_k^j\phi\left(2^jx-k\right); c_k^j=< f^j, \phi_{j,k} >$$

Equation:

$$\mathcal{Q}_j f=\sum_k d_k^j\psi\left(2^jx-k\right); d_k^j=< f^j, \psi_{j,k} >$$

Decomposition algorithm

By combining [link], [link] and [link], we get (see [link]):

Equation:

$$c_k^{j-1} = \sum_l h_{l-2k} c_l^j$$
$$d_k^{j-1} = \sum_l g_{l-2k} c_l^j.$$

Observe that both $\mathbf{c^{j-1}}$ and $\mathbf{d^{j-1}}$ are obtained from $\mathbf{c^j}$ by "moving average" schemes, using the decomposition sequence as "weights", with the exception that these moving averages are sampled only at the even integers. This is called downsampling.

Reconstruction algorithm

In the orthogonal case, the reconstruction algorithm follows easily from the relationships:

Equation:

$$\begin{aligned} c_n^{j+1} &= < f^{j+1}, \phi_{j+1,n} > \\ f^{j+1} &= \mathscr{P}_{j+1} f = \mathscr{P}_j f + \mathscr{Q}_j f \\ &= \sum_k c_k^j \phi_{j,k} + \sum_k d_k^j \psi_{j,k}, \end{aligned}$$

which gives:

Equation:

$$\begin{aligned} c_n^{j+1} &= \sum_k c_k^j < \phi_{j,k}, \phi_{j+1,n} > + \sum_k d_k^j < \psi_{j,k}, \phi_{j+1,n} > \\ &= \sum_k \left[c_k^j h_{n-2k} + d_k^j g_{n-2k} \right]. \end{aligned}$$

Hence $\mathbf{c^{j+1}}$ is obtained from $\mathbf{c^j}$ and $\mathbf{d^j}$ by two moving average.

Mallat's algorithm

In the previous section, we assumed that we knew the coefficients

Equation:

$$c_k^J = < f, \phi_{J,k} >, k \in \mathbb{Z}.$$

The question to ask is: how to compute these coefficients ? In Mallat's algorithm (see [link]), we consider that the finest scale is constituted by the observations$\{Y_k\}_{k=1}^n$ themselves. To use the MRA presented above, these observations must be taken at equispaced points, i.e. we can write that

Equation:

$$\{Y_k\}_{k=1}^n = \left\{ f\left(\frac{k}{n}\right) \right\}_{k=1}^n.$$

Moreover, we assume that n (the number of observations) is a power of 2 : $n = 2^J$, with J denoting the finest level.

Mallat's algorithm is based on the fact that, as j tends to infinity, the support of $\phi_{j,k}$ tends to become smaller and smaller. We have:

Equation:

$$\begin{aligned} \lim_{j\to\infty} \phi_{j,k}(x) \to \delta\left(x - \frac{k}{n}\right) &= 1 \text{ if } x = k/n \\ &= 0 \text{ otherwise.} \end{aligned}$$

Hence Mallat considered that we can compute c_k^J as:

Equation:

$$c_k^J \simeq \int f(x)\delta(x - k/n) = f(k/n) = Y_k.$$

The starting point of this algorithm is thus extremely simple: we just take as value for c_k^J the whole set of observations. Thereafter, having constructed the filters $\{h_k\}$ and $\{g_k\}$ (or, equivalently, having constructed ϕ and ψ), we can compute a fast wavelet transform using the algorithm presented in the previous section.

Approximation of functions with wavelets

In Example from Multiresolution analysis, we saw the Haar wavelet basis. This is the simplest wavelet one can imagine, but its approximation properties are not very good. Indeed, the accuracy of the approximation is somehow related to the regularity of the functions ψ and ϕ. We will show this for different settings: in case the function f is continuous, belongs to a Sobolev or to a Hölder space. But first we introduce the notion of regularity of a MRA.

Regularity of a multiresolution analysis

For wavelet bases (orthonormal or not), there is a link between the regularity of ψ and the number of vanishing moments. More precisely, we have the following:

Note: Let $\{\psi_{j,k}\}$ be an orthonormal basis (ONB) in $L^2(\mathbb{R})$, with $\psi \in C[0.1ex]0.05em1.25ex^{\,m}$, $\psi^{(l)}$ bounded for $l \le m$ and $|\psi(x)| \le C(1+|x|)^{-\alpha}$ for $\alpha > m+1$. Then we have:

Equation:

$$\int x^l \psi(x) dx = 0 \;\text{ for }\; l = 0, 1, \ldots, m.$$

(this describes the “decay in frequency domain”).

This proposition implies the following corollary:

Note: Suppose the $\{\psi_{j,k}\}$ are orthonormal. Then it is impossible that ψ has exponential decay, and that $\psi \in C[0.1ex]0.05em1.25ex^{\infty}$, with all the derivatives bounded, unless $\psi \equiv 0$

This corollary tells us that a trade-off has to be done: we have to choose for exponential (or faster) decay in, **either** time or frequency domain; we cannot have both. We now come to the definition of a $r-$regular MRA (see [link]).

Note: (Meyer, 1990) A MRA is called r-regular ($r \in \mathbb{N}$) when $\phi \in C[0.1ex]0.05em1.25ex^{r}$ and for all $m \in \mathbb{N}$, there exists a constant $C_m > 0$ such that:

Equation:

$$\frac{\partial^{\alpha}\phi\left(x\right)}{\partial x^{\alpha}}| \leq C_m(1+|x|)^{-m} \forall \alpha \leq r$$

If one has a $r-$regular MRA, then the corresponding wavelet $\psi\left(x\right) \in C[0.1ex]0.05em1.25ex^{r}$, satisfies [link] and has r vanishing moments:

Equation:

$$\int x^{k}\psi\left(x\right)dx = 0, k = 0, 1, ..., r.$$

We now have the tools needed to measure the decay of approximation error when the resolution (or the finest level) increases.

Approximation of a continuous function

Note: Let $\{\psi_{j,k}\}$ come from a r-regular MRA. Then, we have, for a continuous function $f \in C0.1ex]0.05em1.25ex\ ^{s}\ (0 < s < r)$the following:

Equation:

$$|< f, \psi_{j,k} > | = O\left(2^{-j(s+1/2)}\right)$$

Note: $(s = 1)$. As $\int \psi(x) = 0$ and $|f\left(x\right) - f\left(k/2^j\right)| \leq C|x - k/2^j|$, ($f$ is Lipschitz continuous), we have:

Equation:

$$\begin{aligned}
|< f, \psi_{j,k} >| &= 2^{j/2} \int \left[f\left(x\right) - f\left(k/2^j\right)\right] \psi\left(2^j x - k\right) dx|, k \in \mathbb{Z} \\
&\leq C 2^{j/2} \int \ x - k/2^j \ ||\psi\left(2^j x - k\right)| dx \\
&= C 2^{-j/2} \int 2^j x - k||\psi\left(2^j x - k\right)| dx \\
&\left(\text{ let } u = 2^j x - k\right) \\
&= 2^{-j(1+1/2)} C \int \ u \ \ \psi\left(u\right)| du < \infty \\
&= O\left(2^{-j(1+1/2)}\right).
\end{aligned}$$

For $s > 1$, we use proposition [link] and we iterate. □

Note that the reverse of [link] is true: [link] entails that f is in C^s.

Note: Under the assumptions of [link], the error of approximation of a function $f \in C[0.1ex]0.05em1.25ex\ ^s$ at scale V_J is given by:

Equation:

$$||f - \sum_k < f, \phi_{J,k} > \phi_{J,k}||_2 = O\left(2^{-Js}\right)$$

Note: Using the exact decomposition of f given by [Equation 30 from Multiresolution Analysis](), one has:

Equation:

$$\begin{aligned} ||f - \mathscr{P}_J f||_2 &= ||\sum_{j \geq J} \sum_k < f, \psi_{j,k} > \psi_{j,k}||_2 \\ &\leq \sum_{j \geq J} C 2^{-j(s+1/2)} ||\sum_k \psi_{j,k}||_2 \end{aligned}$$

Now, if we suppose that $|\psi\left(k\right)| \leq C'(1 + |k|)^{-m}, (m \geq s + 1)$, we obtain:

Equation:

$$||\sum_k \psi_{j,k}||_2 \leq C' 2^{j/2}$$

Putting inequalities [link] and [link] together, we have:
Equation:

$$||f - \mathscr{P}_J f||_2 \leq C'' \sum_{j \geq J} 2^{-js} = C'' \left(1 - 2^{-s}\right)^{-1} 2^{-Js} = O\left(2^{-Js}\right) \square$$

Hence, we verify with this last corollary that, as J increases, the approximation of f becomes more accurate.

Approximation of functions in Sobolev spaces

Let us first recall the definition of weak differentiability, for this notion intervenes in the definition of a Sobolev space.

Note: Let f be a function defined on the real line which is integrable on every bounded interval. If there exists a function g defined on the real line which is integrable on every bounded interval such that:

Equation:

$$\forall x \leq y, \int_x^y g(u) du = f(y) - f(x),$$

then the function f is called weakly differentiable. The function g is defined almost everywhere, is called the weak derivative of f and will be denoted by f'.

Note: A function f is N-times weakly differentiable if it has derivatives $f, f', ..., f^{(N-1)}$ which are continuous and $f^{(N)}$ which is a weak derivative.

We are now able to define Sobolev spaces.

Note: Let $1 \leq p < \infty, m \in \{0, 1, ...\}$. The function $f \in L_p(\mathbb{R})$ belongs to the Sobolev space $W_p^m(\mathbb{R})$ if it is m-times weakly differentiable, and if $f^{(m)} \in L_p(\mathbb{R})$. In particular, $W_p^0(\mathbb{R}) = L_p(\mathbb{R})$.

The approximation properties of wavelet expansions on Sobolev spaces are given, among other, in Härdle **et.al** (see [link]). Suppose we have at our disposal a scaling function ϕ which generates a MRA. The approximation theorem can be stated as follows:

Note: (Approx. in Sobolev space) Let ϕ be a scaling function such that $\{\phi(. - k), k \in \mathbb{Z}\}$ is an ONB and the corresponding spaces V_j are nested. In addition, let ϕ be such that

Equation:

$$\int \left| \phi(x) \right| \left| x \right|^{N+1} dx < \infty,$$

and let at least one of the following assumptions hold:

1. **Equation:**

$$\phi \in W_q^N(\mathbb{R}) \text{ for some } 1 \le q < \infty$$

2. $\int x^n \psi(x) dx = 0, n = 0, 1, ..., N$, where ψ is the mother wavelet associated to ϕ.

Then, if f belongs to the Sobolev space $W_p^{N+1}(\mathbb{R})$, we have:
Equation:

$$||\mathscr{P}_j f - f||_p = O\left(2^{-j(N+1)}\right), \text{ as } j \to \infty,$$

where $\mathscr{P}_j$ is the projection operator onto V_j.

Approximation of functions in Hölder spaces

Here we assume for simplicity that ψ has compact support and is $C[0.1ex]0.05em1.25ex^{1}$ (the formulation of the theorems are slightly different for more general ψ).

Note: If f is Hölder continuous with exponent $\alpha, 0 < \alpha < 1$ at x_0, then

Equation:

$$\max_k \{| < f, \psi_{j,k} > \ dist(x_0, supp(\psi_{j,k}))^{-\alpha}\} = O\left(2^{-j(1/2+\alpha)}\right)$$

The reverse of theorem [link] does not exactly hold: we must modify condition [link] slightly. More precisely, we have the following:

Note: Define, for $\epsilon > 0$, the set

Equation:

$$S\left(x_0, j, \epsilon\right) = \{k \in \mathbb{Z} | supp\left(\psi_{j,k}\right) \cap]x_0 - \epsilon, x_0 + \epsilon\left[\neq \emptyset\right\}.$$

If, for some $\epsilon > 0$ and some $\alpha (0 < \alpha < 1)$,

Equation:

$$\max_{k \in S(x_0, j, \epsilon)} |< f, \psi_{j,k} >| = O\left(2^{-j(1/2+\alpha)}\right),$$

then f is Hölder continuous with exponent α at x_0.